ON GENERALIZED CLIFFORD ALGEBRA $C_4^{(n)}$ AND $GL_q(2;C)$ QUANTUM GROUP

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Abstract

The non commuting matrix elements of matrices from quantum group $GL_q(2;C)$ with $q \equiv \omega$ being the n-th root of unity are given a representation as operators in Hilbert space with help of $C_4^{(n)}$ generalized Clifford algebra generators appropriately tensored with unit 2×2 matrix infinitely many times. Specific properties of such a representation are presented . Relevance of generalized Pauli algebra to azimuthal quantization of angular momentum alà Lévy -Leblond [10] and to polar decomposition of $SU_q(2;C)$ quantum algebra ala Chaichian and Ellinas [6] is also commented.

The case of $q \in C$, $|q| = 1$ may be treated parallelly.

1. Introduction

In 1989 - on the occasion of 70-th birthday of Luigi Radicati - the authors of [1] have written a contribution entitled: *"Properties of Quantum* 2×2 *Matrices"*. They follow there the approach of Ludwig D. Faddeev et all. to the so-called quantum groups [2,3].

Our note deals with the quantum group $GL_q(2;C)$ which is defined [1-6] via imposing quantization relations on the matrix elements of $GL(2;C)$ matrices and correspondingly on $SL(2;C)$ group [1].

The paper [1] of Vocos, Wess and Zumino is at the same time the transparent illustration of the fact that basic representation of a quantum groups posses special properties.

Here we provide a construction of basic representation of the quantum group $GL_q(2;C)$ with $q \equiv \omega \equiv \exp\left\{\frac{2\pi i}{n}\right\}$ being the n-th root of unity - in terms of $C_4^{(n)}$ generalized Clifford algebra generators' tensor products (see [7] and references therein).
Quantization relations [1-6] imposed on the matrix elements of $GL(2;C)$ - result immediately right from the construction.

Specific properties of such a representation are studied following [1]. Generalized Pauli algebras appear naturally in azimuthal quantization of angular momentum alà Lévy -Leblond [10] and also in polar decomposition of the corresponding deformed algebra $SU_q(2;C)$ i.e. "quantum algebra" alà Chaichian and Ellinas [6]. This is also to be commented.

Generalized Pauli algebras appear naturally also in Z_n-Quantum Mechanics [9], in q-deformed Heisenberg algebras [14] , [9] as well as in $q = \omega$ - deformed quantum oscillator [15], [9] as suggested by L. C. Biedenharn in [12].

2. Quantization relations

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;C)$. Then quantization relations [1-6] have the form:

$$\begin{array}{lccc} \{\text{rows}\} & \rightarrow & ab = q\,ba \;\; ; & cd = q\,dc \;\; ; \\ \{\text{columns}\} & \downarrow & ac = q\,ca \;\; ; & bd = q\,db \;\; . \end{array} \tag{2.1}$$

In addition to (2.1) also "diagonal" quantization relations are imposed; these being motivated [1] by the obvious requirement that the product of two quantized matrices (i.e. those satisfying (2.1) and perhaps something else..) should be a matrix of the kind [1-6]. Namely if apart from matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2;C)$ we are given $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL_q(2;C)$ where a', b', c', d' **commute** with a, b, c, d then we expect the matrix $A'' = AA' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ to be also of the same kind i.e. we expect noncommuting quantities a'', b'', c'', d'' to satisfy (2.1) and perhaps something else if necessary. For that to do in addition to (2.1) one requires [1]:

$$\{\text{diagonals}\} \qquad bc = cb\,; \quad ad - da = (q - \frac{1}{q})bc\,. \tag{2.2}$$

If this is accepted then the noncommuting matrix entries of

$$A'' = AA' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

also do satisfy (2.1) and (2.2) with the value of q unchanged. In this connection let us recall [1] that $A^k \equiv \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in GL_{q^k}(2;C)$ i.e. the noncommuting matrix entries of A^k satisfy (2.1) and (2.2) with the value of q^k. (It was proven in [1] that k might be any real number).

Commentary: Conditions (2.1) mean that we have four pairs spanning "q-quantum planes" - four copies of these q-planes.

Question: Is it not then enough to impose (2.1) only?

Perhaps (2.2) might be a technical consequence of the q-geometrical (2.1) requirements‘ representation.

Out of (2.1) and (2.2.) quantization defining commutation relations the authors of [1] derive many interesting properties of $SL_q(2;C)$ quantum group.

It is our aim here to give a simple construction of $GL_q(2;C)$ quantum group and consequently - $SL_q(2;C)$ quantum group for the case of $q \equiv \omega \equiv \exp\left\{\frac{2\pi i}{n}\right\}$. Special cases of such quantum algebras were considered earlier, of course; see [6].

3. The representation of $GL_\omega(2;C)$ elements with help $C_4^{(n)}$ generators.

Let $\{\gamma_i\}_1^4$ be the set of generators of $C_4^{(n)}$ algebra [7] i.e.

$$\gamma_i\gamma_j = \omega\gamma_j\gamma_i\,; \quad i < j\,; \quad \gamma_i^n = id. \quad i,j = 1,2,3,4. \tag{3.1}$$

Let us define now two pairs of tensor products of these generators ($x, y, X, Y \in C$):

$$\begin{array}{cc} a \equiv x\sigma_1 = x\,(\gamma_1 \otimes \gamma_3) \otimes I \otimes I \otimes ...\ ; & b \equiv y\sigma_2 = y\,(\gamma_2 \otimes \gamma_3) \otimes I \otimes I \otimes ...\ ; \\ c \equiv X\Sigma_1 = X\,(\gamma_1 \otimes \gamma_4) \otimes I \otimes I \otimes ...\ ; & d \equiv Y\Sigma_2 = Y\,(\gamma_2 \otimes \gamma_4) \otimes I \otimes I \otimes ...\ . \end{array} \tag{3.2}$$

These are entries of the matrix A from [1] realized as operators in Hilbert space

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = A \equiv \Gamma \equiv \left(\begin{array}{cc} x\sigma_1 & y\sigma_2 \\ X\Sigma_1 & Y\Sigma_2 \end{array}\right)\ ;$$

while the Γ - notation underlines the fact that $\sigma_1, \sigma_2\ ;\ \sigma_1, \Sigma_1$ and $\sigma_2, \Sigma_2\ ;\ \Sigma_1, \Sigma_2$ pairs are obtained from generators of the four corresponding isomorphic copies of $C_2^{(n)}$ generalized Clifford algebra which is called - in this case - a generalized Pauli algebra [7]. (The generalized Pauli algebra [7] $C_2^{(n)}$ generators were already used to provide a representation of the Heisenberg commutation relations for the finite group Z_n in [8] while describing Heisenberg modules for non-commutative two-Torii (see [9] for further connotations).

One now easily verifies that the following commutation relations hold:

$$\begin{array}{lll} \{\text{rows}\} & \sigma_1\sigma_2 = \omega\sigma_2\sigma_1 \quad ; & \Sigma_1\Sigma_2 = \omega\Sigma_2\Sigma_1 \\ \{\text{columns}\} & \sigma_1\Sigma_1 = \omega\Sigma_1\sigma_1 \quad ; & \sigma_2\Sigma_2 = \omega\Sigma_2\sigma_2 \end{array} \tag{3.3}$$

altogether with

$$\sigma_1^n = \Sigma_1^n = \sigma_2^n = \Sigma_2^n = id. \tag{3.4}$$

Thus we have four $\sigma_1, \sigma_2\ ;\ \sigma_1, \Sigma_1$ and $\sigma_2, \Sigma_2\ ;\ \Sigma_1, \Sigma_2$ pairs - four ω-frames of "ω-quantum space" representations.

This is represented by the following pictogram matrix:

$$\begin{pmatrix} \sigma_1 & \rightarrow & \sigma_2 \\ \downarrow & & \downarrow \\ \Sigma_1 & \rightarrow & \Sigma_2 \end{pmatrix} ,$$

showing the order of these ω-commuting entries of quantum matrix Γ in (2.1) commutation relations.

Due to (3.3) a, b, c, d ω-commuting entries defined by (3.2) satisfy (2.1) q-mutation relations "automatically" i.e $ab = q\,ba$; $cd = q\,dc$; $ac = q\,ca$; $bd = q\,db$; $q = \omega$.

Also $bc = cb$ relation is satisfied "automatically" i.e. all is due to the representation .

As for the commutation relations - in order to be complete - we have to consider also the diagonal directions:

$$\begin{pmatrix} \sigma_1 & \rightarrow & \sigma_2 \\ \downarrow & \times & \downarrow \\ \Sigma_1 & \rightarrow & \Sigma_2 \end{pmatrix} .$$

This is an easy task and one readily verifies that

$$\{\text{diagonals}\} \qquad \sigma_2 \bullet \Sigma_1 = \Sigma_1 \bullet \sigma_2 \ ; \ \sigma_1 \bullet \Sigma_2 = \omega^2 \Sigma_2 \bullet \sigma_1 \ . \tag{3.5}$$

In this connection note that $\sigma_1 \bullet \Sigma_2 = \omega^2 \Sigma_2 \bullet \sigma_1$ is equivalent to $\sigma_1 \bullet \Sigma_2 - \Sigma_2 \bullet \sigma_1 = (\omega - \overline{\omega})\, \sigma_2 \Sigma_1 = \left(\omega - \frac{1}{\omega}\right) \sigma_2 \Sigma_1$. At the same time the second commutation relation in (3.5) appears in [1] as a demand in the form:

$$ad - da = \left(q - \frac{1}{q}\right) bc \ ,$$

which for $q = \omega$ might be rewritten as

$$ad - da = (\omega - \overline{\omega}) bc \, . \tag{3.6}$$

In view of (3.2) the requirement (3.6) imposes the following bound on coordinates of the four ω-quantum planes

$$xY = yX \tag{3.7}$$

However because of $\sigma_1 \bullet \Sigma_2 = \omega^2 \Sigma_2 \bullet \sigma_1$, which is equivalent to $\sigma_1 \bullet \Sigma_2 - \Sigma_2 \bullet \sigma_1 = (\omega - \overline{\omega})\, \sigma_2 \Sigma_1$ one may be tempted to replace (see (2.2)) the quantization condition

$$ad - da = (q - \frac{1}{q}) bc \tag{3.8}$$

in the case of $q \equiv \omega \equiv \exp\left\{\frac{2\pi i}{n}\right\}$ by the one resulting from representation of ω-frames i.e.

$$[a, d] = \left(1 - \omega^2\right) \quad \equiv \quad ad = \omega^2 da\,. \tag{3.9}$$

If (3.9) is required instead of (3.8) then we do not have restriction (3.7) on the co-ordinates of the four ω-quantum planes.
However then, one should check whether the product of two quantized matrices (i.e. those satisfying (2.1), $bc = cb$ and (3.9)) gives a matrix satisfying the same q-mutation relations.

Investigation along lines of [1] is plausible and is now being carried out. In all formulas above one may take ω (condition (3.4) being of course rejected)) to be $\omega = \exp\{2\pi i\alpha\}$, with α irrational. The algebras thus generated by (3.3) are no more generalized Pauli algebras of the standard type [7] and they are no more finite dimensional. Nevertheless these algebras deserve to be called infinite dimensional Pauli algebras. In this case (3.3) also implies (3.5) and (3.3) is automatically satisfied in the (3.2) representation.
The representation of $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL_q(2; C)$ where a', b', c', d' **commute** with a, b, c, d is the following

$$\begin{array}{c} a' \equiv x\sigma_1' = xI \otimes I \otimes (\gamma_1 \otimes \gamma_3) \otimes I \otimes I \otimes ... \;; \\ b' \equiv y\sigma_2' = yI \otimes I \otimes (\gamma_2 \otimes \gamma_3) \otimes I \otimes I \otimes ... \;; \\ c' \equiv X\Sigma_1' = XI \otimes I \otimes (\gamma_1 \otimes \gamma_4) \otimes I \otimes I \otimes ... \;; \\ d' \equiv Y\Sigma_2' = YI \otimes I \otimes (\gamma_2 \otimes \gamma_4) \otimes I \otimes I \otimes ... \;. \end{array} \tag{3.2$'$}$$

These are entries of the matrix A' from [1] realized as operators in Hilbert space

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = A' \equiv \Gamma' \equiv \begin{pmatrix} x\sigma_1' & y\sigma_2' \\ X\Sigma_1' & Y\Sigma_2' \end{pmatrix} \;;$$

Thus moving corresponding $(\gamma_i \otimes \gamma_j)$ by step two to the right we obtain others A'', A''' etc. apart from their products $AA'', A''A, ..., A''A'''GL_q(2; C)$.

General conclusion:
The bulk of implications of (2.1) and (2.2) for the quantum group [1-6] in the case of $q \equiv \omega \equiv \exp\left\{\frac{2\pi i}{n}\right\}$ or $\omega = \exp\{2\pi i\alpha\}$, with α irrational *results from the (3.2) representation* of - four ω-frames of "ω-quantum space" *with (3.7) bound on coordinates* being imposed. The possibility of replacing (3.8) by (3.9) deserves to be investigated.

Among the bulk of implications of (2.1) and (2.2) let us here only note [1] that A^k satisfies (2.1) and (2.2) with q^k quantum parameter what results in our case of $q \equiv \omega$ being the n-th root of unity in cyclic arriving to the $q = 1$ case.

4. Interpretation of quantization relations

Now we shall quote and comment interpretations of quantization relations as stated by

$$\begin{array}{lcll} \{\text{rows}\} & \to & ab = q\,ba\ ; & cd = q\,dc\ ; \\ \{\text{columns}\} & \downarrow & ac = q\,ca\ ; & bd = q\,db\ . \end{array} \tag{2.1}$$

$$\{\text{diagonals}\} \qquad bc = cb\,; \quad ad - da = (q - \frac{1}{q})bc\,. \tag{2.2}$$

We shall follow [1] and then [6] keeping in mind that for $q = \omega$ the above quantization relations result from the (3.2) representation with (3.7) bound on coordinates .

Recall first that the quantum or q-deformed determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2;C)$ is defined by

$$D_q = \det{}_q A = ad - qbc = da - q^{-1}bc\ . \tag{4.1}$$

D_q commutes with all elements satisfying quantization relations (2.1) and (2.2). It might be also shown [1] that

$$\det{}_{q^2} A^k = (\det{}_q A)^k \tag{4.2}$$

In our case of $q = \omega$ this means that $\det A^n = (\det{}_\omega A)^n$; hence ω-deformed determinant of quantum matrix A is obtainable form usual determinant of a matrix $B = A^n$ with commuting entries; {note that A^k satisfies (2.1) and (2.2) with q^k}.

Let us also [1] introduce the q-deformed or quantum epsilon matrix

$$\varepsilon_q = \begin{pmatrix} 0 & \frac{1}{\sqrt{q}} \\ -\sqrt{q} & 0 \end{pmatrix} \tag{4.3}$$

Of course $\varepsilon_q^2 = -1$ and as one may check it [1] the quantization relations (2.1) and (2.2) {resulting for $q = \omega$ from the (3.2) representation with (3.7) bound on coordinates}are equivalent to

$$A^T\varepsilon_q A = A\varepsilon_q A^T = D_q\varepsilon_q \tag{4.4}$$

Therefore for $SL_q(2;C)$ quantum group the **quantization relations** may be interpreted as q-deformed or quantum **symplectic conditions** imposed on matrices A under quantization. Let us now come over to the other characterization of quantization relations (2.1) and (2.2) resulting for $q = \omega$ from the (3.2) representation with (3.7) bound on coordinates.

Namely let us interpret matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2;C)$ as endomorphism acting on a quantum plane with points labeled by $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}, \ldots$ etc where $xy = \omega yx\,,\ x'y' = \omega y'x', ..$ etc.

Now if A is supposed to map ω-quantum plane onto the same ω-quantum plane

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{4.5}$$

then quantization relations(2.1) and (2.2) are necessary and sufficient condition for that to be the case [6] , [11].

Coordinates x & y of ω-quantum plane commute with entries of A matrix as it is the case with product of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2;C)$ with $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL_q(2;C)$ where a', b', c', d' **commute** with a, b, c, d.

If entries of A matrix are represented as in (3.2) then there exist infinitely many representations of ω-quantum plane noncommuting coordinates; for example

$$\begin{array}{l} x = xI \otimes I \otimes (\gamma_1 \otimes \gamma_3) \otimes I \otimes I \otimes \ldots\ ; \quad y = yI \otimes I \otimes (\gamma_2 \otimes \gamma_3) \otimes I \otimes I \otimes \ldots\ ; \\ \text{or} \\ x = xI \otimes I \otimes (\gamma_1 \otimes \gamma_4) \otimes I \otimes I \otimes \ldots\ ; \quad y = yI \otimes I \otimes (\gamma_2 \otimes \gamma_4) \otimes I \otimes I \otimes \ldots\ . \end{array} \tag{4.6}$$

or others obtained by moving corresponding $(\gamma_i \otimes \gamma_j)$ to the right.

The observation concerning (4.5) was made in [6] on the occasion of quantum polar decomposition of the algebra of $SU_q(2)$ quantum group. The polar decomposition of the $SU(2)$ group algebra was provided by Lévy-Leblond in his Mexican paper [10]. There he had interpreted such a polar decomposition as a tool for "azimuthal quantization of angular momentum".
It appears that in both cases generalized Pauli matrices (building blocks for all representations of generalized Clifford algebras) appear in the same way [6].

5. Polar decomposition of $SU_q(2;C)$ and $SU(2;C)$ groups' algebras

The standard basis of Lie algebra $su(2) = so(3)$ is well known to be represented by:

$$
\begin{aligned}
J_3 &= \sum_{m=-j}^{j} m\,|jm\rangle \quad ; \\
J_+ &= \sum_{m=-j}^{j} \sqrt{(j-m)\,(j+m+1)}\,|j\,(m+1)\rangle\ \langle jm| \quad ; \\
J_- &= \sum_{m=-j}^{j} \sqrt{(j+m)\,(j-m+1)}\,|j\,(m+1)\rangle\ \langle jm| \quad .
\end{aligned}
\tag{5.1}
$$

In [12] Biedenharn proposed a new realization of quantum group $SU_q\,(2)$ and in order to realize generators of a q-deformation $U_q(su(2))$ of the universal enveloping algebra of the Lie algebra $su(2)$ he defined a pair of mutual commuting q-harmonic oscillator systems (alà Jordan-Schwinger approach to $su(2)$ Lie algebra).

At the same time in [13] Mac Farlane had also provided us with q-oscillator description of $SU_q\,(2)$ (alà Jordan-Schwinger approach to $su(2)$ Lie algebra). The generators of a q-deformation $U_q(su(2))$ of the universal enveloping algebra of the Lie algebra $su(2)$ (called by physicists "generators of the quantum group $SU_q\,(2)$" - which is not a group!) are obtained from (5.1) by one of several possible q-deformations. In Biedenharn's and Mac Farlane's case one uses the following q-deformation of numbers and operators:

$$
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \quad . \tag{5.2}
$$

Thus q-deformed (5.1) representation of generators now reads

$$
\begin{aligned}
J_3 &= \sum_{m=-j}^{j} m\,|j,m\rangle_q \quad ; \\
J_+ &= \sum_{m=-j}^{j} \sqrt{[j-m]_q\,[j+m+1]_q}\,|j,(m+1)\rangle_{q\ q}\,\langle j,m| \quad ; \\
J_- &= \sum_{m=-j}^{j} \sqrt{[j+m]_q\,[j-m+1]_q}\,|j,(m+1)\rangle_{q\ q}\,\langle j,m| \quad ,
\end{aligned}
\tag{5.3}
$$

where

$$
|j,m\rangle_q = |j+m\rangle_q\,|j-m\rangle_q = \frac{a_{1q}^{+\ j+m}\ a_{2q}^{+\ j-m}}{[j+m]_q!\,[j-m]_q!}\,|0\rangle_q \tag{5.4}
$$

and a_{1q}^+ , a_{2q}^+ represent two mutually commuting creation operators of q-quantum harmonic oscillators. Corresponding commutation relations of the generators of a q-deformation $U_q(su(2))$ of the universal enveloping algebra of the Lie algebra $su(2)$ are of the familiar though now q-deformed form [12], [13], [6]:

$$
[J_3, J_+] = J_+ \quad ; \quad [J_3, J_-] = -J_- \quad ; \quad [J_+, J_-] = [2J_3]_q \quad . \tag{5.5}
$$

From (5.3) one may derive [6] the polar decomposition of the generators J_+, J_-:

$$J_+ = \sqrt{J_+J_-}\sigma_1^{-1} = \sigma_1^{-1}\sqrt{J_-J_+} \qquad J_- = \sqrt{J_+J_-}\sigma_1 = \sigma_1\sqrt{J_-J_+}\ , \tag{5.6}$$

where σ_1 is the first of the two generators of generalized Pauli algebra [7], [9]

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = (n \times n) \tag{5.7}$$

The second generator σ_2 has been also used in [6] in order to remark on relevance of such a pair σ_1 , σ_2 to $GL_\omega(2; C)$ properties. It is to be noted here that the polar decomposition for undeformed $su(2) = so(3)$ algebra of undeformed quantum angular momentum had been performed already by Lévy -Leblond in his Mexican paper [10]. There he had interpreted such a polar decomposition as the "azimuthal quantization of angular momentum". And it should be also noted here - following the authors of [6] that - what we know as - generalized Pauli algebra appears in q-deformed and in undeformed case of polar decomposition in the same way (5.6).

Neither Lévy -Leblond nor the authors of [6] had realized that they are dealing with generalized Pauli algebra [9]. These has been realized afterwards by T. S. Santhanam [15].

In the notation of [9] and earlier papers quoted there

$$\sigma_2 \equiv U = \omega^Q = \exp\left\{\frac{2\pi i}{n}Q\right\} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix} \tag{5.8}$$

where

$$Q = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 \end{pmatrix}\ ; \tag{5.9}$$

$$\sigma_1 \equiv V = \omega^P = \exp\left\{\frac{2\pi i}{n}P\right\} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{5.10}$$

where

$$P = S^{+}QS = (P_{\alpha,\kappa}) \tag{5.11}$$

$$P_{\alpha,\kappa} = \begin{cases} 0 & \alpha = \kappa \\ \left[\overline{\omega}^{(\alpha-\kappa)} - 1\right]^{-1} & \alpha \neq \kappa \end{cases}$$

and

$$S = \left(< \widetilde{k}|l >\right) = \frac{1}{\sqrt{n}} \left(\omega^{kl}\right)_{k,l\in Z_n} \tag{5.12}$$

is the Sylvester matrix.

Formulas (5.8) - (5.12) contain the main information on quantum kinematics of the finite dimensional quantum mechanics. Here we interpret polar decomposition of quantum angular momentum algebra $su(2) = so(3)$ formalism as a model of finite dimensional quantum mechanics with the classical phase space being the torus $Z_n \times Z_n$ (see [9] and references therein) . This possibility was already considered by Weyl in [16].

"Azimuthal quantization of angular momentum" was interpreted afterwards as the finite dimensional quantum mechanics by Santhanam et. all [15].

The considerations of this section allow us to hope to elaborate soon more on the q-deformed finite dimensional quantum mechanics treated as an interpretation of q-deformed $su(2)$ algebra of q-deformed angular momentum (for example by Schwinger method).

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